# Intermediate Mathematical Olympiad
## Maclaurin paper
**Monday 15 March 2021**

© 2021 UK Mathematics Trust

*supported by* [XTX] **Overleaf**

*England & Wales: Year 11*
*Scotland: S4*
*Northern Ireland: Year 12*

These problems are meant to be challenging! The earlier questions tend to be easier; later questions tend to be more demanding.

Do not hurry, but spend time working carefully on one question before attempting another.

Try to finish whole questions even if you cannot do many: you will have done well if you hand in full solutions to two or more questions.

You may wish to work in rough first, then set out your final solution with clear explanations and proofs.

## Instructions

1. Do not open the paper until the invigilator tells you to do so.
2. Time allowed: **2 hours**.
3. The use of blank or lined paper for rough working, rulers and compasses is allowed; **squared paper, calculators and protractors are forbidden**.
4. Write on one side of the paper only and start each question on a fresh sheet.
5. Write your participant ID and question number neatly in the top left corner of each page and arrange them with your cover sheet on top, so that your teacher can easily upload them to the marking platform. **Do not hand in rough work**.
6. Your answers should be fully simplified, and exact. They may contain symbols such as $\pi$, fractions, or square roots, if appropriate, but not decimal approximations.
7. You should give full written solutions, including mathematical reasons as to why your method is correct. Just stating an answer, even a correct one, will earn you very few marks; also, incomplete or poorly presented solutions will not receive full marks.

Enquiries about the Intermediate Mathematical Olympiad should be sent to:

*UK Mathematics Trust, School of Mathematics, University of Leeds, Leeds LS2 9JT*

☎ 0113 365 1121     enquiry@ukmt.org.uk     www.ukmt.org.uk

◇ *Do not hurry, but spend time working carefully on one question before attempting another.*

◇ *Try to finish whole questions even if you cannot do many.*

◇ *You will have done well if you hand in full solutions to two or more questions.*

◇ *Your answers should be fully simplified, and exact. They may contain symbols such as π, fractions, or square roots, if appropriate, but* not *decimal approximations.*

◇ *Give full written solutions, including mathematical reasons as to why your method is correct.*

◇ *Just stating an answer, even a correct one, will earn you very few marks.*

◇ *Incomplete or poorly presented solutions will not receive full marks.*

◇ *Do* not *hand in rough work.*

**1.** Solve the pair of simultaneous equations

$$x^2 - 2xy = 1,$$
$$5x^2 - 2xy + 2y^2 = 5.$$

**2.** The 12 points in the first diagram below are to be joined in pairs by 6 line segments that pass through the interior of the triangle. One example is shown in the second diagram. In how many ways can this be done?
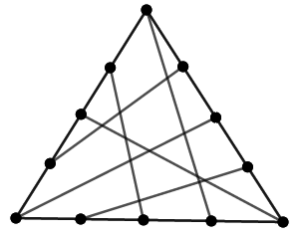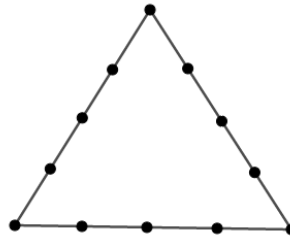
**3.** The diagram shows a triangle $ABC$. A circle touching $AB$ at $B$ and passing through $C$ cuts the line $AC$ at $P$. A second circle touching $AC$ at $C$ and passing through $B$ cuts the line $AB$ at $Q$.

Prove that $\dfrac{AP}{AQ} = \left(\dfrac{AB}{AC}\right)^3$.
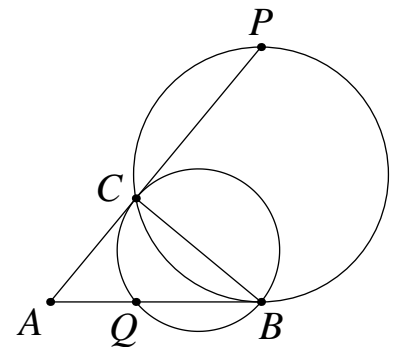
**4.** A sequence of integers $a_1, a_2, a_3, \ldots$ is defined by

$$a_1 = k,$$
$$a_{n+1} = a_n + 8n \text{ for all integers } n \geq 1.$$

Find all values of $k$ such that every term in the sequence is a square.

**5.** A triangular playground has sides, in metres, measuring 7, 24 and 25. Inside the playground, a lawn is designed so that the distance from each point on the edge of the lawn to the nearest side is 2 metres. What is the area of the lawn?

**6.** A cat and a mouse occupy the top right and bottom left cells respectively of an $m \times n$ rectangular grid, where $m, n > 1$. Each second they both move diagonally one cell.

For which pairs $(m, n)$ is it possible for the cat and the mouse to occupy the same cell at the same time?

**Note:** For every pair $(m, n)$ you must either prove that it is impossible for the cat and the mouse to occupy the same cell at the same time, or explain why there is a sequence of moves that ends with the cat and the mouse occupying the same cell at the same time.